Signal and Noise Problems

M. Kac

American Mathematical Monthly, Volume 61, Issue 7, Part 2: Proceedings of the Symposium on Special Topics in Applied Mathematics (Aug. - Sep., 1954), 23-26.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28195408%2F09%2961%3A7%3C23%3ASANP%3E2.0.CO%3B2-K

SIGNAL AND NOISE PROBLEMS

M. KAC, Cornell University

In this brief report I shall limit myself to a cursory discussion of several problems chosen on grounds of personal predilection and with the view of showing how "pure" and "applied" mathematics can intermingle with profit to both.

1. The problem of detection. This is a purely statistical problem which can be formulated as follows: Let $x(t)$ be a stationary random process (noise) and $s(t)$ a signal. An observer receives a record $y(t)$ which is either $x(t)$ or $x(t)+s(t)$ and is to decide whether the signal is present or not.

In the simplest case, when $s(t)$ is periodic with period θ and a finite number of observations is made at times $t_1, t_1+\theta, \cdots, t_1+(n-1)\theta$ and if furthermore θ is long compared with the correlation time of the noise (*i.e.*, $x(t_1+j\theta), j=1, 2, \cdots, n-1$, are independent), the problem can be solved by the use of the Neyman-Pearson theory. A detailed account can be found in [1]. More recently various authors [2] considered the sequential approach to the problem. In its general form the problem clearly belongs to decision theory and it may be hoped that here the theories of the late A. Wald will play an important part.

2. Spectra and correlation. The relation between the power spectrum $A(\omega)$ and the correlation function $\rho(t)$,

$$\rho(t) = \int_{-\infty}^{\infty} A(\omega) \cos \omega t \, d\omega, \tag{2.1}$$

is a standard tool in the theory of noise. It is often desirable to consider processes for which

$$\rho(t) = 0 \qquad |t| > A. \tag{2.2}$$

The relations (2.1) and (2.2) combined with the fact that $A(\omega) \geqq 0$ induce strong restrictions on $\rho(t)$. One can show, for instance, [3] that if n is an integer

$$\left|\rho\left(\frac{A}{n}\right)\right| \leqq \cos \frac{\pi}{n+1} \rho(0), \tag{2.3}$$

the constant being the best possible.

3. Integral equations. In the theory of radio receivers with square-law detectors one is led to the problem of finding the distribution function of an expression

$$\int_0^{\infty} K(\tau) \{ x^2(t-\tau) + y^2(t-\tau) \} dt, \tag{3.1}$$

where $x(t)$ and $y(t)$ are independent, stationary Gaussian processes with the same correlation function $\rho(t)$.

The characteristic function of the distribution function of (3.1) is given by (see [4])

$$D^{-1}(i\xi),$$

where $D(\lambda)$ is the Fredholm determinant of the integral equation

$$\lambda \int_0^\infty \rho(s - t)K(t)\phi(t)dt = \phi(s). \tag{3.2}$$

Integral equations of the form (3.1) appear in other branches of pure and applied mathematics and it is at least amusing to contemplate solving them by building a corresponding receiver and determining the distribution function experimentally.

4. Zeros of random functions. This is an extremely interesting and difficult problem where much further work needs to be done. If $x(t)$ is a stationary Gaussian process with power spectrum $A(\omega)$, the average number of zeros per unit time is given by Rice's formula [5]:

$$2\left\{\frac{\int_{-\infty}^{\infty} \omega^2 A(\omega)d\omega}{\int_{-\infty}^{\infty} A(\omega)d\omega}\right\}^{1/2}. \tag{4.1}$$

The fact that by counting zeros one can obtain information about the spectrum is in itself of great practical interest. It has been, for instance, applied to turbulence by H. W. Liepmann and his group at the California Institute of Technology. A more detailed study of the distribution of zeros of random functions for even the simplest processes encounters great analytical difficulties. A closely related problem is the following:

Let

$$f(t) = \sum_1^n a_k \cos 2\pi(\lambda_k t + \phi_k) \tag{4.2}$$

and assume that the frequencies λ_k are rationally independent. Let $N(T, a)$ be the number of roots of

$$f(t) = a$$

in $0 \leqq t \leqq T$.

It can then be demonstrated [6] that

$$\lim_{T\to\infty} \frac{N(T, a)}{T} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int \frac{\cos a\xi}{\eta^2}\left\{\prod_{k=1}^n J_0(a_k\eta) - \prod_{k=1}^n J_0(a_k\sqrt{\xi^2 + \lambda_k\eta^2})\right\} d\xi d\eta. \tag{4.3}$$

This is the counterpart of (4.1) and can be interpreted by saying that the average distance between consecutive a-values of $f(t)$ is given by the inverse of the expression on the righthand side of (4.3). If one asks for the average of the *square* of the distance between consecutive a-values one runs into difficulties which, at least at present, appear insurmountable. This is already true for the simplest case

$$f(t) = a_1 \cos \lambda_1 t + a_2 \cos \lambda_2 t$$

(except for special values of a_1, a_2 and a). An interesting application of (4.3) to the theory of unimolecular reaction rates was made recently by N. B. Slater [7]. In conclusion let me mention another related problem, this time with no practical implications.

Consider a polynomial of degree n

$$\sum_{k=0}^{n} X_k t^k \tag{4.4}$$

whose coefficients are independent, normally distributed, random variables each having mean 0 and variance 1. It is then easy to show [8] that the average number of real roots of (4.3) is asymptotically

$$\frac{2}{\pi} \log n. \tag{4.5}$$

Moreover, a tedious but rather elementary calculation shows that the standard deviation about the mean is of lower order and consequently it is very rarely that a random algebraic equation of high degree will have a number of real roots which is significantly different from (4.5).

These conclusions remain valid for a much wider class of independent random variables [9] but proofs become enormously more tedious. For the simplest case

$$\text{Prob. } \{X_k = 1\} = \text{Prob. } \{X_k = -1\} = \tfrac{1}{2}$$

the proof that (4.5) is still asymptotically the average number of real roots is lacking!

The present exposition was naturally permeated with probabilistic considerations. But perhaps it is not too idle and inappropriate to contemplate here the possibility of a statistical approach to various questions in pure mathematics. As an example let me consider the following question: how good is the classical Descartes' rule of signs?

As applied to an individual algebraic equation the question is largely meaningless. Interpreted statistically it can be properly formulated and answered. The average number of changes of sign in (4.4) is $n/2$ (this is to be compared with $\pi^{-1} \log n$ which is the average number of positive real roots of (4.4)).

However, one can do better. If one considers the polynomial

$$\left(\sum_0^n t^k\right)\left(\sum_0^n X_k t^k\right),$$

the number of real roots is the same as for (4.4); the average number of changes of sign can now be shown to be of the order $C\sqrt{n}$. This is about the best one can do, and yet $C\sqrt{n}$ is still so far from the correct order $\pi^{-1}\log n$ that we must conclude that Descartes' rule of sign is extremely unlikely to give a good estimate for equations of high degree.

Bibliography

1. J. L. Lawson and G. E. Uhlenbeck, Threshold Signals, McGraw-Hill, 1950.

2. See *e.g.*, W. C. Fox, Signal detectability, Eng. Rev. Inst., Univ. of Michigan. Here references to previous work, especially that of D. Middleton, are given.

3. R. P. Boas, Jr. and M. Kac, Inequalities for Fourier transforms of positive functions. Duke Math. J., vol. 12, 1945, pp. 189–206. (See also the errata *ibid.* 15, 1948, pp. 105–109.)

4. M. Kac and A. J. F. Siegert, On the theory of noise in radio receivers with square law rectifiers. J. Appl. Phys., vol. 18, 1947, pp. 383–397.

5. S. O. Rice, Mathematical analysis of random noise. Bell. Tech. J., vol. 23, 1944, pp. 282–332.

6. M. Kac, On the distribution of values of trigonometric polynomials with linearly independent frequencies. Am. J. Math., vol. 65, 1943, pp. 609–615. See also a recent paper by N. B. Slater: Some formulas of P. Stein and others concerning trigonometrical sums. Proc. Cam. Phil. Soc., vol. 50, 1950, pp. 33–39.

7. N. B. Slater, Aspects of a theory of unimolecular reaction rates. Proc. Roy. Soc. A194, 1948, pp. 112–131.

8. M. Kac, On the average number of real roots of random algebraic equations. Bull. Amer. Math. Soc., vol. 49, 1943, pp. 314–320.

9. M. Kac, On the average number of real roots of random algebraic equations II, Proc. London Math. Soc., vol. 50, 1949, pp. 390–408.